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BASIC FUNCTIONAL AND GEOMETRIC INEQUALITIES FOR THE FRACTIONAL ORDER OPERATORS ON HOMOGENOUS LIE GROUPS

ABSTRACT

of the PhD thesis for the degree of
doctor of Philosophy (PhD) in the specialty
«6D060100-Mathematics»

The relevance of the research topic. The first mathematicians who study of subelliptic analysis on the Heisenberg group were Folland and Stein, who consistently created a generalisation of the analysis for more general stratified groups. And it can also be noted that Rothschild and Stein generalised these results for general vector fields satisfying the Hormander's condition. We can say that these results were published in the famous book by Folland and Stein which laid the anisotropic analysis. And it is worth noting that homogeneous Lie group is nilpotent.

The history of fractional calculus originates from the works of Riemann and Liouville. And in these works, the concepts of the fractional integral were introduced for the first time. Along with integer derivatives, the concept of a fractional derivative was introduced, which was named after Riemann and Liouville. Then, Hadamard in his works, he introduced a different definition of the fractional derivative. And it is also worth noting that Caputo also introduced the definition of a fractional derivative that in a particular case can be equal to the Riemann-Liouville derivative. These operators are non-local operators. Note that these fractional derivatives are one-dimensional operators. For the multidimensional case, the concept of a multidimensional fractional Laplacian is introduced via Laplacian's symbol. It is worth noting that fractional calculus is currently a rapidly developing mathematical field. The main aim of this dissertation is to combine non-commutative analysis on groups and fractional calculus.

Nowadays, functional and geometric inequalities on Lie groups are currently a rapidly developing field of mathematics. Many nonlinear differential equations of problems of mechanics and problems of physics to which the global solvability of problems is proved through functional inequalities. It means, one of the most important tool to study PDE is the functional inequalities. For example, integer order multi-dimensional Hardy inequality demonstrates the following inequality:

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \le \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \quad 1 < p < n, \ \ \forall u \in C_0^\infty(\mathbb{R}^n),$$

where $|\cdot|$ is the Euclidean distance and constant $\left(\frac{p}{n-p}\right)^p$ is a sharp. This inequality has applications in a lot of areas of mathematics, for example in spectral theory. Also, by this inequality we can show Heisenberg-Pauli uncertainly principle, which has application in quantum theory. Firstly, on group settings Hardy inequality was obtained by Garofalo and Lanconelli on Heisenberg group.

The aim of the PhD thesis is to study the fractional subelliptic functional and geometric inequalities on homogenous groups such as Hardy, Sobolev Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities. Also, we study reverse inequalities. We study reverse integral Hardy inequality on metric measure space. As consequence, we show reverse integral Hardy inequality on homogeneous groups, hyperbolic space and Cartan-Hadamard manifolds. Then we show application of functional inequalities in nonlinear PDE on homogeneous groups. In addition, we establish one-dimensional fractional functional inequalities for the Caputo, Riemann-Liouville and Hadamard derivatives. Then we give some applications of the one dimensional inequalities.

Objectives of the study:

- Direct functional inequalities;
- Reverse functional inequalities;
- Applications of these inequalities to nonlinear PDE;
- One-dimensional functional inequalities on interval.

Object of the PhD thesis is the proof of the main fractional functional inequalities on homogeneous Lie groups, namely, Hardy, Sobolev Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities, reverse integral Hardy inequality on metric measure space, its consequence on homogeneous groups, hyperbolic space and Cartan-Hadamard manifolds, reverse Hardy-Littlewood-Sobolev, reverse Stein-Weiss, improved reverse Stein-Weiss inequalities on homogeneous Lie groups, its application in nonlinear PDE and one-dimensional functional inequalities.

The methods of scientific research. For research on this PhD thesis, the classical methods of calculus, fractional calculus, the theory of homogeneous Lie groups, theory of PDE, along with new ideas of mathematical sciences are used.

Scientific novelty of the work. The following new results were obtained in the PhD thesis:

- *Direct inequalities.* In this chapter, For the direct inequalities we obtain fractional Hardy, Sobolev, Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, logarithmic inequalities, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups. Also, we obtain integer order Sobolev-Folland-Stein inequality on stratified groups.

- *Reverse inequalities*. For the reverse inequalities, we prove reverse integral Hardy inequalities with parameters $q < 0,\ p \in (0,1)$ and $-\infty < q \leq p < 0$. Also, we show reverse integral Hardy inequalities on homogeneous Lie groups, hypebolic space and Cartan-Hadamard manifolds with $q < 0,\ p \in (0,1)$. As a consequences, we show reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved version Stein-Weiss inequalities for the cases $q < 0,\ p \in (0,1)$ and $-\infty < q \leq p < 0$. In addition, we obtained the reverse Hardy, L^p-Sobolev and L^p-Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups.

- *Applications*. In this chapter, we show some applications of these inequalities in linear and nonlinear PDE on homogeneous groups. Firstly, we obtain Lyapunov inequality for fractional p-sub-Laplacian equation and systems and show application in spectral theory. Also, we show existence of the weak solutions for some nonlinear PDE on homogeneous Lie groups. Then we show blow-up results to the heat equation with logarithmic nonlinearity and fractional sub-Laplacian on homogeneous Lie groups. Then we obtain blow-up result for viscoelastic equation on stratified Lie groups. Then we obtain Fujita and Kato's exponent to the heat and wave Rockland equation (system) on graded Lie groups.

- *Appendix*. In this chapter, we consider one-dimensional functional inequalities on Euclidean case. We establish fractional Hardy, Poincar\'{e} type, Gagliardo-Nirenberg type and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of these inequalities for the first eigenvalue and we show de La Vall\'{e}e Poussin-type inequality for fractional elliptic boundary value problem.

Theoretical and practical significance of the results. This study has mainly fundamental character, makes a valuable contribution to the development of theory of functional analysis on nilpotent Lie groups and theory of differential equations, and promotes raising the image of the Republic of Kazakhstan in the scientific world. Obtained results will be applied to solving various problems in theoretical physics.

Publications. On the topic of the thesis 14 papers were published, including 3 publications in scientific journals included in the list recommended by the Committee on the Control of Education and Science of the MES RK for publication of the main scientific results of scientific activities, 7 publications in a high-ranking scientific journal, i in the Thomson database Reuters or Scopus database, 4 publications in materials of international conferences,**.**

The results on the topic of the thesis were published in the following papers:

Publication in the high-ranking scientific journals

1 A. Kassymov, M. Ruzhansky and D.Suragan. Fractional logarithmic inequalities and blow-up results with logarithmic nonlinearity on homogeneous groups. Nonlinear Differ. Equ. Appl., 27:7, 2020. (Scopus, Web of Science, Q1);

2 A. Kassymov, M. Ruzhansky and D. Suragan. Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups.
Integral Transform. Spec. Funct., 30(8):643--655, 2019. (Scopus, Web of Science, Q2);

3 A. Kassymov and D. Suragan. Existence of solutions for p-sub-Laplacians with nonlinear sources on the Heisenberg group. Complex Variables and Elliptic Equations, dx.doi:10.1080/17476933.2020.1731737, 2020. (Scopus, Web of Science, Q2);

4 A. Kassymov, B. Torebek and N. Tokmagambetov. Nonexistence Results for the Hyperbolic-Type Equations on Graded Lie Groups. Bulletin of the Malaysian Mathematical Sciences Society, doi:10.1007/s40840-020-00919-6, 2020, (Scopus, Web of Science, Q2);

5 Bekbolat B., Kassymov A., Tokmagambetov N. Blow-up of Solutions of Nonlinear Heat Equation with Hypoelliptic Operators on Graded Lie Groups. Complex Analysis and Operator Theory, 13(7):3347-3357, 2019. (Scopus, Web of Science, Q2);

6 A. Kassymov and D. Suragan. Fractional Hardy-Sobolev inequalities and existence results for frational sub-Laplacians. Journal of Mathematical Sciences, to appear. (Scopus, Q3);

7 A. Kassymov and D. Suragan. Lyapunov-type inequalities for the fractional p-sub-Laplacian. Advances in Operator Theory, 1-18, doi:10.1007/s43036-019-00037-6, 2020. (Scopus, Web of Science);

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1 A. Kassymov and Suragan D. An analogue of the fractional Sobolev inequality on the homogenous Lie groups. Mathematical Journal, 18(1):99-110, 2018;

2 A. Kassymov and Suragan D. Reversed Hardy–Littlewood–Sobolev inequality on homogeneous Lie groups. Kazakh Mathematical Journal, 19(1):50-57, 2019;

3 A. Kassymov. Blow-up of solutions for nonlinear pseudo-parabolic Rockland equation on graded Lie groups. Kazakh Mathematical Journal, 19(3):89-100, 2019.

Publications in materials of international conferences

1 A. Kassymov. On the S-number inequalities of triangular cylinders for the heat operator. Traditional International April scientic conference in honor of the Science Day. -April 10, 2018 - Almaty. - P.109.

2 A. Kassymov, A. Kashkynbayev and Suragan D. Non-blow-up and blow-up results to heat equations with logarithmic nonlinearity on stratified groups. «Traditional International April math conference in honor Day of Science Workers of the Republic of Kazakhstan, dedicated to the 1150th anniversary of Abu Nasyr al-Farabi and 75th anniversary of the Institute of Mathematics and mathematical Modeling», p.45, 2020.

3 A. Kassymov, A. Kashkynbayev and Suragan D.. Blow-up results for viscoelastic wave equations with damping terms on stratified groups. «Traditional International April math conference in honor Day of Science Workers of the Republic of Kazakhstan, dedicated to the 1150th anniversary of Abu Nasyr al-Farabi and 75th anniversary of the Institute of Mathematics and mathematical Modeling», p.46, 2020.

4 A. Kassymov and Suragan D. Cordoba-Cordoba type inequality on homogenous Lie groups. Fourth International Conference on Analysis and Applied Mathematics. - September 6-9, 2018- North Cyprus- P. 109.

The structure and scope of the thesis. The PhD thesis includes a title page, content, introduction, prelimenaries, 3 chapters, appendix, conclusion and list of references, consisting of 129 titles. The total volume of the thesis is 205 pages.

The main content of the thesis. The introduction reflects the substantiation of the relevance of the topic of the dissertation, the purpose, object, subject, research objectives, the substantiation of the scientific novelty of the work, its theoretical and practical significance, the scientific principles to be defended, the number of publications available.

In Chapter 3, we develop theory of the fractional functional and geometric inequalities on homogeneous Lie groups. We obtain the fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and its logarithmic fractional inequalities which is even new on Euclidean case. For the Riesz potential (or a fractional integral), we get the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which means boundedness of the Riesz operator in $L^q - L^p$ spaces. Also, we obtain the Stein-Weiss inequality (or a radially weighted Hardy-Littlewood-Sobolev inequality) for the Riesz potential. In addition, we show integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups.

In Chapter 4, we study a question of the reverse functional inequalities. Firstly, we start to study reverse integral Hardy inequality on metric measure space. We note that, in the Ruzhanky and Verma's, the authors introduced polar

decomposition on metric measure space, which is play a key role in their proof. In this chapter, we obtain reverse integral Hardy inequality on metric measure space with parameters $q < 0$ and $p \in (0,1)$. As consequences, we get integral reverse Hardy inequality on on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifoldse with parameters $q < 0$ and $p \in (0,1)$. Also, we show integral reverse Hardy inequality on metric measure space with parameters $\infty < q \leq p < 0$ and as a consequences we show reverse integral Hardy inequality on homogeneous Lie groups. Then we obtain the reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$ and $p \in (0,1)$. Also, we obtain the reverse Hardy-Littlewood-Sobolev, Stein-Weiss type and improved Stein-Weiss type inequalities with parameters $\infty < q \leq p < 0$, which is even new in Euclidean settings. In addition, we obtain the reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups.

In Chapter 5, we give applications of the functional inequalities in PDE. Firstly, we obtain Lyapunov inequalities for the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. As a application of Lyapunov's inequality, we give lower estimate of the first eigenvalue of the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. Then, we show existence of the weak solution for the nonlinear equation with the p-sub-Laplacian on the Heisenberg and stratified groups. Also, we show existence of the weak solution for the nonlinear equation with the fractional sub-Laplacian and Hardy potential on homogeneous Lie groups and multiplicity of the weak solution with first stratum Hardy potential on Heisenberg and stratified groups. Then we discuss blow-up results for heat equation with fractional sub-Laplacian and logarithmic nonlinearity on homogeneous Lie groups and for heat equation with sub-Laplacian and logarithmic nonlinearity on stratified group. Also, we show blow-up results for viscoelastic equations with sub-Laplacian on stratified groups, heat and wave Rockland equations on graded groups.

In Appendix, we consider one-dimensional functional inequalities on Euclidean case. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg type and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of this inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem.